Equilibrium Matching and Termination

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Abstract

We construct an equilibrium model of the labor market that combines the theory of equilibrium search and matching and the theory of dynamic contracting. Jobs are dynamic contracts. Equilibrium layoffs and retirements are terminations of dynamic contracts. Transitions from unemployment to new jobs are modelled as a process of matching and bargaining. We then calibrate the model to the U.S. economy to study worker turnover, compensation dynamics and distribution. We show that the model can generate equilibrium wage dispersions similar to that in the data. Hornstein, Krusell and Violante (2006) argue that standard search matching models can generate only a very small differential between the average wage and the lowest wage paid in the labor market.

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1 Introduction

We construct an equilibrium model of the labor market that combines the theory of equilibrium search and matching and the theory of dynamic contracting. Jobs are dynamic contracts. Equilibrium layoffs and retirements are terminations of optimal dynamic contracts, as in Wang (2005). Transitions from unemployment to new jobs are modelled as a process of matching and bargaining, as in Mortensen and Pissarides (1994). Matched workers and firms bargain over the values of the optimal contract to each party, and then the dynamics of the optimal contract will take them to a state of termination. Firms enter freely into the market to endogenously determine the number of jobs in the economy.

The standard Mortensen-Pissarides equilibrium matching model of the labor market is built around two key mechanisms: a matching-bargaining mechanism that sets the worker-firm pair up for an employment relationship, and a dynamic but exogenous process of match productivity then provides an engine for job separation. An important extension of the standard match model is Moscarini (2005), who puts the model of Jovanovic (1979) into the Mortensen-Pissarides framework to model separation as a process of learning about the productivity of the match, and allows the match to be dissolved once the perceived match productivity is sufficiently low.

We take a dynamic contract point of view to modelling equilibrium job separation in the Mortensen-Pissarides model. In this paper, we let the worker and firm enter into an optimal dynamic contract upon a match, and job separation is then modelled as the termination of the dynamic contract. In our environment of moral hazard, termination is used as an incentive device to induce worker efforts and as a way of minimizing the cost of worker compensation. Workers that produce a sequence of bad outputs become too poor to motivate, and workers who produce a sequence of good outputs become too expensive to compensate and motivate, as in Spear and Wang (2005). Following terminations, workers are free to go back to the labor market to seek new matches, or choose to stay temporarily or permanently out of the labor market. This generates equilibrium flows between employment and unemployment and into retirements.

Thus job separation is a purely endogenous process in our model, motivated by the dynamic provision of incentives and risk sharing. Workers and firms are homogeneous in our model. Matches are identical: they all operate the same production function in all periods. Termination occurs not because the technology of the match has evolved to be sufficiently poor as in Mortensen and Pissarides (1994), or it is found out to be sufficiently bad as in Moscarini (2005). Termination occurs because the economic relationship that evolved endogenously around the fixed match technology has become too costly for the parties to maintain.

As in Wang (2005), modelling job separation as termination of a dynamic contract in a moral hazard environment allows us to model simultaneously employment, unemployment, and retirements, and to determine endogenously the size and composition of the economy's not-in-the-labor-force (NLF), as well as the flows into and out of NLF. This is important, not only for the explanation of the evolution of NLF itself, but also necessary for providing a more coherent and complete view of the stocks and flows of the labor market. Existing labor market data, especially the recently available Current Population Survey, have shown significant flows of workers among all three states of the labor market: employment, unemployment, and not in the labor force, as documented in several recent researches including Fallick and Fleischman (2004), Nagypal (2005), Shimer (2005b).

Most existing models in the search-matching literature of the labor market focus on the interaction between employment and unemployment (e.g., Mortensen and Pissarides (1997), Shimer (2005), Moscarini (2005), Nagypal (2005)), without modelling explicily the state of non-participation in the labor force. Sun-Bin Kim (2001) and Moscarini (2003) are exceptions. In both papers though, an additional source of worker heterogeneity is introduced into the Mortensen-Pissarises framework in order to generate flows into retirement. [1] In our model, unemployment and non-participation are motivated by the same information friction in a clean model environment with homogeneous workers and matches. Workers choose to leave the labor market because they have accumulated enough wealth to make their efforts too expensive for the firms and to make them more expensive to motivate (Spear and Wang (2005)). Unlike in Lazear (1979), there are no life cycles in our model either.

A critical feature of the Mortensen-Passarides model is that workers and firms cannot commit to long-term relationships, and wages are bargained sequentially upon the realizations of the current match productivity. In our model, firms can fully commit to any long-term contracts, although workers are allowed to quit an ongoing employment relationship if it offers a value lower than the workers' outside alternative. Bargaining occurs only once, before the contract is signed, and is over the total values of the optimal contract to the parties. Wages are state contingent compensation payments to the worker that are dictated by the optimal structure of the optimal contract, not bargained directly and repeatedly each period.

In an important recent study, Hornstein, Krusell and Violante (2006) argue that standard search matching models can generate only a very small, 3.6%, differential between the average wage and the lowest wage paid in the labor market, whereas the oberved Mm ratio–the ratio between the average wage and lowest wage paid– is at least twenty times larger than what the model observes. As the paper explains, "The short unemployment durations, as in the U.S. data, reveal that agents in the model do not find it worthwhile to wait because frictional wage inequality is tiny. The message of search theory is that "good things come to those who wait", so if the wait is short, it must be that good things are not likely to happen." (page 9.) Hornstein, Krusell and Violante further show that the extensions to the standard search and matching models can only modestly improve their performance on accounting for the overved

[1] The productivity of a match depends on a match specific variable, as well as a non-match-specific variable that captures the ability of the worker. The values of both variables must be learnt.

Mm ratio.

Our model is capable of generating much larger wage dispersions than standard search and matching models do. In the version of the model that is calibrated to the U.S. data, the computed Mm ratio is 24.5, similar to what Hornstein, Krusell and Violante observe in the U.S. data. In our model, wage dispersion is driven by the provision of intertemporal incentives and intertemporal risk sharing. Wages of homogenous workers who start with the same initial expected utility fan out over time as their outputs follow a stchastic process. In our model, workers who produce a sequence of high outputs will see their wages increase over time, and workers that produce a sequence of low outputs will see their wages decrease over time. This effect of the dynamic contracting on distribution was first discussed in Green (1987) and Atkeson and Lucas (1991). This paper puts this mechanism to workn in a search/matching framework.

The model is presented in the next section. In Sections 3 and 4 we formulate and analyze a stationary equilibrium of the model. We then calibrate the model to the U.S. economy in Section 5 to study the optimal dynamic contract, the stocks and flows of the labor market, and worker compensation dynamics.

2 Model

Let time be denoted $t = 1, 2, \cdots$ The model economy has one consumption good, and is populated by one unit of homogeneous workers. Workers survive into the next period with a constant probability $\Delta \in (0, 1)$. At the beginning of each period, $1 - \Delta$ units workers are born so the measure of workers in each period is constant at one. Workers have the following preferences:

$$E_0 \sum_{t=1}^{\infty} (\beta\Delta)^t [u(c_t) - \phi(a_t)],$$

where $\beta \in (0, 1)$ is the worker's discount factor, $u : \mathbb{R}_+ \to \mathbb{R}$ denotes the worker's utility function, c_t his consumption; $\phi : \mathbb{R}_+ \to \mathbb{R}$ denotes the worker's disutility function, a_t his effort. We make the following assumptions on u and ϕ: $u(\cdot)$ is continuous, differentiable, and strictly concave; $\phi(\cdot)$ is continuous, differentiable, and strictly convex.

The model economy is also populated with a large measure of identical firms. Firms maximize expected discounted profits, and they discount future profits using a constant discount factor $1/(1 + r)$, where $r > 0$. In any given period, some firms are in the market, the rest not. Firms are allowed to freely enter or exit the market and so the measure of the firms that are in the market, γ, is an endogenous variable. Firms in the market must be matched with a worker in order to produce. A matched pair of firm and worker creates a job.

In any given period, the total measure of matches formed in the labor market is equal to

$$M(\eta_A, \gamma - \eta_E),$$

where η_A is the measure of the unemployed workers (non-employed and actively looking for a job) in the labor market, and η_E is the measure of the workers that are currently employed when the labor market opens, and hence $\gamma - \eta_E$ is the measure of vacant (recruiting) firms in the labor market. Throughout the paper, we assume

$$0 \leq M(\eta_A, \gamma - \eta_E) < \min\{\eta_A, \gamma - \eta_E\},$$

so there is always a positive measure of workers *and* firms that are not matched.

A firm that fails to find a match could either exit the market or to operate as a vacant firm in the rest of the period, waiting for the labor market to open next period. We follow the literature to assume that a vacant firm must incur a fixed cost $c_0(\geq 0)$ in order to stay open to job applications.

The matched firm and worker Nash bargain over a dynamic employment contract for the worker. This dynamic contract specifies a history contingent rule for compensating and terminating the worker. Once they agree on a specific contract, this contract cannot be renegotiated in any future periods.

Production then takes place immediately after a contract is agreed on. In each period, the employed worker produces a random output $\theta \in \{\theta_1, \cdots, \theta_n\}$ with probabilities $\{\pi_1(a), \cdots, \pi_n(a)\}$, where $a \in A$ is the worker's effort, $\pi_i : A \to [0, 1]$, and $A \subseteq \mathbb{R}_+$ is the set of possible effort levels. We assume that the worker's effort is not observable to the firm. That is, there is moral hazard.

There is a risk free asset in the model: for each unit of the good invested in this asset, it returns $(1 + r)$ units of consumption next period. To avoid introducing additional information asymmetry, we assume that all investments in this asset are public information and transferable among workers and firms. Workers also have access to a competitive insurance market where one unit of consumption in the current period can be exchanged for $1/\Delta$ units of consumption in the next period conditional on the worker's survival in the next period.

As part of the model's physical environment, we make three assumptions about the contracts that are feasible between the worker and the firm. First, contracts are subject to a non-negativity constraint that requires that compensation to the worker be non-negative. Second, once the worker and the firm agree on a contract, they can commit to not renegotiating the continuations of the contract in all future dates. Third, firms can fully commit to the terms of a long-term contract, whereas workers' commitment to a long-term contract is limited: workers are free to leave an ongoing long-term contract anytime there is a better outside value. Forth, severance payments must be made in lump-sum amounts to the worker immediately upon termination. Once an employment relationship is terminated, no further interactions between the firm and the worker are feasible.

3 Equilibrium

In this section, we formulate the economy's stationary equilibrium. We first describe the economy's aggregate state variables. We then describe the optimization problems that workers and firms face, taking the aggregate states as given. Finally, we require that the aggregate states and individual optimization be consistent with each other, and that the firms in the market be making zero profits.

3.1 The aggregate states

At the beginning of each period, the state of the model economy is characterized by the following aggregate state variables

$$\Sigma = \{(S, \mu_N, \eta_N),\ (X, \mu_E, \eta_E), \gamma\},$$

where γ is the measure of firms in the market; $\eta_N \in (1-\Delta, 1)$ is the measure of the non-employed workers, these workers are distributed over the set $S = \mathbb{R}_+$ according to the distribution function $\mu_N : S \to [0,1]$, where S is the set of feasible amounts of assets each worker can hold. The scalar $\eta_E \in (0,1)$ denotes the the measure of workers that are employed at the beginning of a period. The employed workers are distributed over

$$X \equiv \left[\frac{u(0)-\phi(0)}{1-\beta\Delta}, \frac{u(\infty)}{1-\beta\Delta}\right) \equiv [V_{min}, V_{max}),$$

the set of all possible expected utilities of the employed workers at the beginning of a period, the distribution function being $\mu_E : X \to [0,1]$. Clearly, η_N and η_E must satisfy

$$\eta_N + \eta_E = 1. \tag{1}$$

3.2 Optimization

Conditional on Σ, an optimal solution to the firm's and the worker's optimization problems is a duple

$$\sigma \equiv \left\{ \begin{array}{c} \Omega(V), a(V), (c_i(V), V_i(V)), V \in \Phi \\ \overline{U}; U(V), V \in \Phi \\ (\eta_A, S_A, \mu_A), (\eta_I, S_I, \mu_I) \\ v(s), s \in \mathbb{R}_+; V_m(s), s \in S_A, V_n(s), s \in \mathbb{R}_+ \end{array} \right\}$$

where

(i) The tuple $\{\Omega(V), a(V), \Omega(V), c_i(V), V_i(V), V \in \Phi\}$ is the dynamic contract for the currently employed worker. Here we follow Green (1987) and Spear and Srivastava (1987) to use the worker's beginning of period expected utility as a state variable to summarize history. The set $\Phi \subseteq X$ is the state space. This is the set of all expected utilities of the employed worker that can be delivered by a (sub-game perfect) feasible and incentive compatible contract. Note that Φ is an endogenous variable of the model. Then, for all $V \in \Phi$, $a(V)$ denotes the worker's recommended effort in the current period, $\Omega(V)$ denotes the set of worker's output realizations in which the worker is retained, and outside which the worker is terminated. c_i and V_i are, respectively, the worker's current compensation (consumption) and next period utility if his current output is θ_i.

(ii) For any $V \in \Phi$, $U(V)$ denotes the value of a firm who currently employs a worker with expected utility V. $\overline{U} \in \mathbb{R}$ is the value of a vacant firm: a firm that is free to hire a new worker at the beginning of a period, before the market opens.

(iii) The set $S_A \subseteq \mathbb{R}_+$ denotes the set of assets of the non-employed workers that will choose to participate in the current labor market, and the set $S_I = \mathbb{R}_+ - S_A$ denotes the set assets of the non-employed workers that will not participate in the current labor market.Note that the worker could choose to stay out of the labor market for a number of periods and then reenter. The scalars η_A and η_I denote, respectively, the numbers of the non-employed workers that belong to the sets S_A and S_I, respectively. We have $\eta_A + \eta_I = \eta_N$. Finally, $\mu_A : S_A \to [0, 1]$ and $\mu_I : S_I \to [0, 1]$, respectively, are the probability distributions of the non-employed workers who are in the labor market and those who are not.

(iv) For any given s, $v(s)$ denotes maximized the beginning of period (before the labor market opens) expected utility of an non-employed worker with assets s; $V_n(s)$ denotes the expected utility of this worker if is not matched with a firm (either he chose not to participate in the market ($s \in S_I$), or he went to the market ($s \in S_A$) but failed to find a match); $V_m(s)$ denotes the bargained expected utility of the worker conditional on (a) he chose to go to the market (*i.e.*, $s \in S_A$) and (b) he is matched with a firm.

Let λ denote the fraction of the vacant firms to obtain a match is

$$\lambda = \frac{M(\eta_A, \gamma - \eta_E)}{\gamma - \eta_E}, \tag{2}$$

and let rho denote the fraction of unemployed workers to transition to employment (ratio of hiring out of the pool of the unemployed),

$$\rho = \frac{M(\eta_A, \gamma - \eta_E)}{\eta_A}. \tag{3}$$

The restriction we put on the matching function M ensures $0 < \lambda, \rho < 1$.

Definition 1 *We say that σ is an optimal solution to the firm and the worker's optimization problem, conditional upon a given set of the market's states Σ and the implied λ and ρ, if it satisfies the following conditions (I) to (IV).*

Condition (I)

$$\overline{U} = \lambda \int_{S_A} (U(V_m(s)) + s) d\mu_A(s) + (1-\lambda)\beta\overline{U} - c_0 \tag{4}$$

Condition (II) *For all $V \in \Phi$,*

$$\begin{aligned} U(V) &= \max_{\{\Omega, c_i, V_i, a\}} \sum_{i \notin \Omega} \pi_i(a) \left[\theta_i - c_i + \beta\Delta\left(\overline{U} - v^{-1}(V_i)\right)\right] \\ &+ \sum_{i \in \Omega} \pi_i(a) \left[\theta_i - c_i + \beta\Delta U(V_i)\right] + \beta(1-\Delta)\overline{U} \end{aligned} \tag{5}$$

subject to (6)-(10) where

$$V = \sum_{i=1}^{n} \pi_i(a)[u(c_i) + \beta\Delta V_i] - \phi(a) \tag{6}$$

$$a = \arg\max_{a'} \left(\sum_{i=1}^{n} \pi_i(a')[u(c_i) + \beta\Delta V_i] - \phi(a') \right) \tag{7}$$

$$\Omega \subseteq \Theta, \tag{8}$$

$$c_i \geq 0, \ \forall i \tag{9}$$

$$V_i \in \Phi, \forall i \in \Omega \tag{10}$$

$$V_i \in v(\mathbb{R}_+), \forall i \notin \Omega \tag{11}$$

$$V_i \geq v(0), \ \forall i \tag{12}$$

where the function $v : \mathbb{R}_+ \to R$, its inverse v^{-1}, which is to be shown to exist later in the paper, and the value of $v(0)$, will be given in Condition (IV);

Condition (III) *The set Φ of all payoffs for an employed worker that can be generated by a feasible and incentive compatible contract is the largest self-generating set of the mapping $B : 2^X \to 2^X$ defined by: $\forall \Phi' \subseteq X$,*

$$B(\Phi') \equiv \{V \in X | \exists \{\Omega, a, c_i, V_i\} \ s.t. \ (6) - (9), (11), (12), \text{ and } V_i \in \Phi'\}. \tag{13}$$

Condition (IV) *The non-employed-worker's porobtlem about whether to enter the labor market and the related values are described by*

$$S_A \equiv \{s \in \mathbb{R}_+ : \exists V \in \Phi, V \geq V_n(s) \text{ such that } U(V) + s \geq \beta\overline{U}\}, \tag{14}$$

$$S_I \equiv [0, +\infty) \diagdown S_A \tag{15}$$

where

$$V_n(s) = \max_{0 \leq c \leq s} \left\{ u(c) - \phi(0) + \beta \Delta v \left(\frac{1+r}{\Delta}(s - c) \right) \right\} \; \forall s \in [0, +\infty), \tag{16}$$

$$V_m(s) = \arg \max_{V \in \Phi, V \geq V_n(s), U(V) + s - \beta\overline{U} \geq 0} \left(U(V) + s - \beta\overline{U} \right)^{\omega} (V - V_n(s))^{1-\omega}, \; \forall s \in S_A, \tag{17}$$

$$\forall s \geq 0, \; v(s) = \begin{cases} \rho V_m(s) + (1-\rho) V_n(s), \; \text{if } s \in S_A \\ V_n(s), \; \text{if } s \in S_I \end{cases} \tag{18}$$

Conditions (I)-(IV) formulate the set of Bellman equations for the values of the firms and the workers, along with the optimal strategies.

In equation (4), with probability λ the vacant firm is matched with an unemployed worker whose assets s is drawn randomly from the distribution μ_A. Once matched, the worker gives his assets s to the firm, and the firm gives the worker an employment contract that promises the worker expected utility $V_m(s)$. This $V_m(s)$ is the solution to the Nash bargaining problem to be formulated in equation (17).

Implicitly in equation (4) is the assumption that assets are transferable between the worker and the firm. Suppose assets are not freely transferable, then an additional state variable will be needed for the recursive formulation of the dynamic contract. We leave this possibility for future work.

In equation (5), $v^{-1}(V_i)$ is the cost to the firm of letting the worker leave the firm with a promised utility equal to V_i. Here v^{-1} is the inverse of the worker's value function v which, in turn, is defined in equation (18). That is, in order to guarantee that the worker obtains a level of expected utility equal to V_i, the firm must make a severance payment to the worker in the amount $v^{-1}(V_i)$.

Observe that at this stage, it is not clear why the inverse function v^{-1}, and the function V_m are well defined. In the next section, we will show that the function v is well defined, continuous, and strictly increasing over its domain $\mathbb{R}_+$, and hence its inverse exists and is monotonic. We will also show that $V_m(s)$ is well defined for each $s \in S_A$.

Constraints (10) and (11) require that the expected utility V_i promised to the worker must be feasible. Specifically, if the worker is retained, then the promised utility must be achievable by a sub-game perfect feasible and incentive compatible contract; if the worker is terminated, then the expected utility the worker receives must be supportable by a feasible severance payment s.

The constraint $V_i \geq v(0)$ is a self-enforcing constraint. Under this constraint, the worker will not have an incentive to leave the contract in all ex post state of the world. This constraint is not needed if we assume full commitment.

Condition (III) provides a Bellman equation for the state space of the recursive optimal contract. This follows Abreu, Pearce and Stachetti (1990) and Wang (1997).

Equation (14) defines the set of non-employed workers that are in the market, S_A. The conditions imposed on a $s \in S_A$ say that, in order for a non-employed worker to be willing to participate in the labor force, there must be a feasible and incentive compatible contract to make the worker and the matched firm both better off should they form a match.

Equation (16) describes the optimization problem for the nonemployed worker who is not matched with a firm (either he was in the market but failed to form a match, or he chose not to participate in the labor market).

Equation (17) lays out the problem of Nash bargaining between a worker and a firm who are matched. The parameter $\omega \in (0,1)$ is the exogenously given bargaining weight. Since in each period each firm and each worker can find at most one match, $\beta\overline{U}$ is the firm's reservation utility, and $V_n(s)$ is the worker's.

Note that implicit in Equation (17) is the assumption that (a) the Nash bargaining problem has a solution and (b) the solution is unique. Proposition 4 in the next section will verify that this assumption is satisfied.

Equation (18) describes the non-employed worker's decision about whether or not to participate in the current labor market. Note that since the worker does not incur any costs being in the labor market, we make the assumption that workers who do have a zero probability to be hired will voluntarily stay out of the labor market.

3.3 Equilibrium

Definition 2 *A stationary equilibrium of the model is a tuple $\{\Sigma, \lambda, \rho, \sigma\}$ that satisfies the following conditions:*

(i) λ and ρ are given by (2) and (3).

(ii) Conditional on Σ, ρ and λ, σ solves the worker and the firm's optimization problem.

(iii) Σ is generated by σ and is stationary.

(iv) Free entry of firms into the market ensures

$$\overline{U} = 0. \tag{19}$$

4 Analysis

In this section, we analyze the Bellman equations in Definition 1 that jointly characterize the worker and the firm's optimization problems. We begin with a set of useful observations. Observe first that

$$V_m(s) \geq v(s) \geq V_n(s), \ \forall s \in S_A. \tag{20}$$

This holds because the definitions of S_A and $V_m(s)$ imply $V_m(s) \geq V_n(s)$ and that $v(s)$ is a convex combination of $V_m(s)$ and $V_n(s)$ for all $s \in S_A$. Notice next that

$$V_n(0) = u(0) - \phi(0) + \beta\Delta v(0).$$

Notice also that

$$v(0) \geq V_{\min}.$$

This holds because the non-employed worker with $s = 0$ can always choose to stay out of the labor market permanently to obtain V_{min}. Notice therefore

$$v(0) \geq V_n(0) = u(0) - \phi(0) + \beta\Delta v(0) \geq V_{min}. \tag{21}$$

We now proceed with the analysis. Notice that the function $v(\cdot)$ plays a central role in the definition of the Bellman equation. First, $v(\cdot)$ is the only link between the firm's optimization problem, which is defined by equations (4) to (12), and the rest of the equations that define the worker's problem. Second, as we will show, if $v(\cdot)$ is well defined and continuous, then the worker's other value functions $V_n(\cdot)$ and $V_m(\cdot)$ are well defined and continuous. So the strategy of our analysis is to formulate the function $v(\cdot)$ as a fixed point of a contraction mapping on a space of bounded and continuous function, and then use the contraction mapping theorem to obtain that $v(\cdot)$ is uniquely defined and continuous.

Proposition 3 *Suppose the function v is continuous. Then*

$$\Phi = \left[u(0) - \phi(0) + \beta\Delta v(0), \ \frac{u(\infty)}{1 - \beta\Delta}\right).$$

Assumption 1 The value function $U : \Phi \to \mathbb{R}$ is continuous and concave.

This assumption is reasonable, for the continuity and concavity of U could always be obtained through randomization over employment contracts if necessary. See Athey and Bagwell (2001).

Notice that $U(V) \to -\infty$ as $V \to u(\infty)/(1 - \beta\Delta)$. This holds because, independent of the contract used, the expected profits of a firm are bounded from above while the cost of delivering V to the worker goes to infinity as V goes to $u(\infty)/(1 - \beta\Delta)$. With this and Assumption 1, let

$$V^* = \max \arg\max_{V \in \Phi} U(V). \tag{22}$$

This V^* exists, is unique, and has the following interpretation: If the firm is free to offer any expected utility to a newly hired worker, V^* is the highest starting expected utility of the worker that achives the highest value for the firm.

Proposition 4 *For any $s \in S_A$, if $v(s)$ is well defined, then the solution to the bargaining problem (17) exists and is unique.*

Assumption 2 There exists a feasible and incentive compatible one-period contract σ_0 that offers the worker expected utility $V_0 \geq u(0) - \phi(0)$ and the firm expected profit $\Pi(V_0) > 0$.

Lemma 5 *Under Assumption 2,* $0 \in S_A$.

Lemma 5 implies directly that

$$U(V^*) > \beta\overline{U}. \tag{23}$$

Lemma 6 *Suppose the function v is well defined and continuous. Then (i) V_n is a well defined, continuous, and strictly increasing function on $\mathbb{R}_+$; (ii) V_m is well defined, continuous, and increasing on S_A; and (iii) v is strictly increasing on $\mathbb{R}_+$.*

Theorem 7 *The function v is well defined and continuous.*

Following Lemma 6 and Theorem 7, we have

Corollary 8 *(i) The functions V_n and V_m are well defined and continuous. (ii) V_n and v are strictly increasing functions. (iii) The function V_m is a weakly increasing.*

By Theorem 7, we also have

$$v(\mathbb{R}_+) = [v(0), V_{max}). \tag{24}$$

With this, and given (21), we can rewrite constraints (10)-(12) as

$$V_i \in [v(0), V_{max}). \tag{25}$$

This leads to

Proposition 9 *With the optimal contract, $i \in \Omega$ if and if $U(V_i) > \overline{U} - v^{-1}(V_i)$*

This result is intuitive, it says that the worker is retained if the value of retention is greater than the value of termination. Notice that since in equilibrium $\overline{U} = 0$, the above proposition permits the firm to retain a worker that has a negative value to the firm, as long as the value of terminating them is even lower.

Proposition 10 *Under Assumption 1, there exists $\overline{s} > 0$ such that $[0, \overline{s}] \subseteq S_A$.*

Proposition 11 *Suppose a newly-terminated worker with expected utility V goes back to the labor market immediately [i.e., $v^{-1}(V) \in S_A$]. Then either $\overline{U} > 0$, or there exists $V' \in \Phi$ such that $V' > V$ and $U(V') > U(V)$.*

Given that $U(V)$ is concave, in equilibrium with $\overline{U} = 0$, in order for a worker to go back to the labor market immediately after termination, his expected utility must be sufficiently low, lower than V^*. In other words, a newly terminated worker is unemployed if he is terminated from the left hand side of the firm's value function. Worker who are terminated from the right hand side of the firm's value function will stay out of the labor force for at least one period.

5 Quantitative Analysis

In this section, we calibrate our model to the U.S. data, analyze it numerically, and show that our model could do a better job accounting for the observed wage dispersion than standard search/matching models.

5.1 Parameterization and Calibration

We set the time period to be one month. We set the discount rate to be $r = 0.00417$ to obtain an annual interest rate of 5%. We then set the worker's discount factor to be $\beta = 1/(1 + r)$. We set $\Delta = 0.99815$ so the worker's expected lifetime is 45 years.

We set the worker's utility function to be

$$u(c) - \phi(a) = \log(\rho_0 + \rho_1 c) - a^2,$$

where ρ_0 is normalized to 1 and $\rho_1 > 0$.

We set $n = 2$ so output can be low (θ_1) or high (θ_2). We assume

$$\pi_1(a) = \exp(-\psi a),\ \pi_2(a) = 1 - \exp(-\psi a),\ \forall a \geq 0,$$

where $\psi > 0$. We follow the literature to assume a Cobb-Douglas matching function so that

$$M(\eta_A, \gamma - \eta_E) = \alpha_0 {\eta_A}^{\alpha} (\gamma - \eta_E)^{1-\alpha}$$

The above parameterization leaves us with the following parameters for the calibration of the model:

$$\theta_1,\ \theta_2,\ \rho_1, \psi, \alpha_0, c_0, \gamma.$$

We target a measure of unemployed workers equal to 0.0342, a measure of employed workers equal to 0.6336, and a measure of those not in the labor force equal to 0.3320. These values are derived from The Current Population Survey (CPS) which provides monthly time series data on employment, unemployment and not-in-the-labor-force, for the period between January 1994 and December 2003. These target measures imply an unemployment rate of 5.12%, and a labor force participation rate of 66.78%.

We target a job finding probability of 28.3%, following Fallick and Fleischman (2004). We follow the literature to set $\alpha = 0.6$. The literature reports a value of α between 0.5 and 0.7 (Blanchard and Diamond (1989), Petrongolo and Pissarides (2001)).

Davis, Faberman and Haltiwanger (2007) reports a job opening rate of 3.4% for the period from December 2000 to January 2005.[2] Using this information, we choose the value of α_0 to generate a job finding probability (fraction of the unemployed to flow into employment) of 28.3%:

$$\alpha_0 \left(\frac{\gamma - \eta_E}{\eta_A} \right)^{1-0.6} = \alpha_0 \left(\frac{0.034 * 0.6336}{0.0342} \right)^{1-0.6} = 0.283$$

which gives us $\alpha_0 = 0.3405$. In addition, given that

$$\text{job opening rate} = \frac{\gamma - \text{employment}}{\text{employment}}$$

we obtain $\gamma = 1.034 \times 0.6338 = 0.6551$.

We follow Shimer (2005) to set $\omega = 0.4$ (Hosios 1990). We could alternatively set $\omega = 0.5$ without significantly change the calibration outcome.

We are now left with five free parameters $\theta_1, \theta_2, \psi, \rho_1, c_0$, and we choose their values to target 6 (essentially 5) measures of the U.S. data: the measures of employment (E), unemployment (U), non-participation (N); the rate of flow from employment to unemployment, the job finding probability (rate of flow from unemployment to employment), and the job opening rate (vacancies as a fraction of employment).

The following table gives the values of the parameters chosen.

Parameter	Value
θ_1	-0.5000
θ_2	2.5000
ψ	0.6386
ρ_1	1.2771
c_0	0.0096

The following table compares the calibrated model with data.

[2] Their measure is based on the Job Openings and Labor Turnover Survey (JOLTS).

Variable	Data	Model
fraction of employment	0.6336	0.6317
fraction of unemployment	0.0342	0.0350
fraction of not in the labor force	0.3320	0.3333
job opening rate	3.4%	3.7%
E to U probability	1.3%	1.26%
U to E probability	28.3%	29.1%

The model does a good job matching the targets. Note that conditional on employment, which is an independent target to match, the job openning rate essentially measures the stock of vacancies in the economy.

The U.S. data shows a large flow from unemployment to not-in-the-labor-force, reflecting the movements of discouraged workers, and the movements from unemployment to education. Our model lacks a channel for the flow from unemployment to NLF.

The U.S. data also shows a significant flow of workers from not-in-the-labor-force to employment. This reflects the fact that, in practice, firms search not only among workers that are unemployed, but also among workers that are not in the labor force. This mechanism is missing in our model. In the model, workers must be actively looking for jobs before being matched with a firm.

Finally, notice that the flows from employment to unemployment and from employment to not-in-the-labor-force are much smaller in the model than in the data. These are not surprising. In the data, a large fraction of the transitions from employment to not in the labor force are due to life-cycle reasons, or younger workers quitting the labor force to obtain higher education. (??) These are not in our model. In the data, the flow from not in the labor force to unemployment reflects perhaps the movements of the previously discouraged workers and the young workers who enter the labor market after finishing education.

5.2 Equilibrium

Figure 1 depicts the firm's net gains from retaining (rather than terminating) the worker as a function of the worker's expected utility. In order to deliver a given level of expected utility V to the worker, the firm's net profits are $U(V)$ if it retains the worker and $\overline{U} - v^{-1}(V)$ if it terminates the worker, and in equilibrium $\overline{U} = 0$. The value of the difference is shown in Figure 1. Obviously, termination is optimal if and only the value of V is sufficiently small or sufficiently large. This is consistent with Wang (2005).

Figure 2 depicts the law of motion for the employed worker's expected utility as a function of his current output. The worker's expected utility is higer (lower) next period if his current is higher (lower) this period.

Figure 4 shows the (deterministic) law of motion for the worker's assets: $s_{t+1} - s_t$ as a function of s_t. There is critical asset level above which the non-employed worker chooses not to enter the labor market. For sufficiently high asset levels, there is not a non-negative Nash surplus between the worker and the firm.

The stationary distributions of employed workers and non-employed workers (unemployed plus not-in-in-the-labor-force) are shown in Figures 5 and 6, respectively. There is clearly a significant amount of welfare dispersion among the employed workers.

At each point in time, looking forward each employed worker faces a stochastic number of periods over which to remain employed. Figure 8 depicts the distribution of the duration of the current job for a worker with four different level of starting expected utilities. Obviously, the worker who has an expected utility that is neither too low nor too higher will longer on this current job on average.

In equilibrium, a worker who leaves his job with an expected utility above the upper bound of the retention interval will not go back to the labor market immediately, a worker who leaves his job with an expected utility below the lower bound of the retention interval will go back to labor market right away. Hence, the former consists of the employment to not-in-the-labor-force transition, and the latter consists of the employment to unemployment transition.

Furthermore, each worker is born without any saving. As a result, his first job is characterized by a contract delivering relatively low expected utility. It takes time for him to establish a good record, and in turn be promised a relatively high expected utility.

Figure 9 is based on simulation and shows that the probability to transition from employment to unemployment decreaes with the age of the worker, while the probability to transition from employment to not-in-the-labor-force is increasing with the age of the worker. These are consistent with findings in Nagypal (2005). [3].

5.3 Wage Dispersion

Hornstein, Krusell and Violante (2006) show that standard search matching models can generate only a very small, 3.6%, differential between the average wage and the lowest wage paid in the labor market, whereas the oberved Mm ratio–the ratio between the average wage and lowest wage paid– is at least twenty times larger than what the model observes. Hornstein, Krusell and Violante further show that the extensions to the standard search and matching models can only modestly improve their performance on accounting for the overved Mm ratio. As HKV argue, the logic of the search/matching model implies that a higher wage dispersion is associated with longer unemployment durations or a smaller probability of finding employment for

[3] In Nagypal (2005), the probability to transition from employment to not-in-the-labor-force for younger workers is unusually high which might be explained by the higher education admission.

the unemployed. Given that unemployment durations are typically short in the data, wage dispersion cannot be large in the model. [4]

This logic of the search/matching model does not apply in our model. In our model, wage dispersion is driven by the provision of intertemporal incentives and risk sharing. Wages of homogenous workers who start with the same initial expected utility fan out over time as their outputs follow a stchastic process. In our model, workers who produce a high output not only receive a higher wage in the current period, but also will see their future utilities and wages increased. Likewise, and workers that produce a low outputs will receive lower wages in the current period and in the future. [5]

Our model is capable of generating much larger wage dispersions than standard search and matching models do. In the version of the model that is calibrated to the U.S. data, the computed average wage is 0.4071, and the lowest wage paid is 0.0166, and the Mm ratio is 24.5, similar to what Hornstein, Krusell and Violante observe in the U.S. data.

Suppose we use the average wage of the workers in the lowest wage percentile as the minimum wage in the calculation, then the computed Mm ratio is 13.89. Even if we use the average wage of the workers in the 5th wage percentile as the minimum wage in the calculation, the computed Mm ratio is 5.32, much larger than what the search/matching models permit. Note that our model generates the same job finding probability for the unemployed, and hence the same average unemployment duration, as the calibrated search/matching models do.

6 Conclusion

In this paper, we have studied an equilibrium labor market model which modifies the Mortensen-Pissarides framework by taking a dynamic contract approach to jobs and job separations. The dynamic contract approach we take is based on a standard information friction: moral hazard. Dynamic contracting under moral hazard generates equilibrium worker flows from emoloyment to unemployment, and to non-labor-force particiation. Matching and bargaining bring unemployed workers to employment. As in the data, in the model average wages increase with worker tenure, and on average workers who have stayed longer with the firm face lower layoff probabilities. Our model offers an important advantage over standard search and matching models: we have shown quantitatively that our model generates wage dispersions that are similar to those observed in the data while standard search-matching models cannot.

[4] As the paper explains, "The short unemployment durations, as in the U.S. data, reveal that agents in the model do not find it worthwhile to wait because frictional wage inequality is tiny. The message of search theory is that "good things come to those who wait", so if the wait is short, it must be that good things are not likely to happen." (page 9.)

[5] The effect of the dynamic contracting on distribution was first discussed in Green (1987) and Atkeson and Lucas (1991).

7 Appendix

Proof of Proposition 3 We first show that $\left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{(1-\beta\Delta)}\right) \subseteq \Phi$. It suffices to show that the set $\left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$ is self-generating. Consider the following recursive contract: For all $V \in \left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$, let

$$\Omega(V)=\Theta,\ a(V)=0,\ c_i(V)=x(V),\ V_i(V)=v(0)+y(V),$$

where $x(V)$ and $y(V)$ satisfy $x(V)\geq 0$, $y(V)\geq 0$, and

$$V=u(x(V))-\phi(0)+\beta\Delta[v(0)+y(V)]. \tag{26}$$

This recursive contract obviously satisfies the constraints (6)-(9), the condition $V_i(V) \in \left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$, and the constraints $V_i(V) \in v(R_+)$ and $V_i(V) \geq v(0)$; and it generates all $V \in \left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$. This proves that $\left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$ is self-generating.

Next, we show that $\Phi \subseteq \left[u(0)-\phi(0)+\beta\Delta v(0),\ \frac{u(\infty)}{1-\beta\Delta}\right)$. We need only show that there does not exist $V \in \Phi$ such that $V < u(0)-\phi(0)+\beta\Delta v(0)$. This is true because the worker can always choose $a=0$ to obtain an expected utility greater than or equal to $u(0)-\phi(0)+\beta\Delta v(0)$, independent of the contract offered. Q.E.D.

Proof of Proposition 4 (ii) Let $s \in S_A$. We show that the solution to the following optimization problem exists and is unique:

$$\max O(V)\ s.t.\ V\in\Phi,\ V\geq V_n(s),\ U(V)+s-\beta\overline{U}\geq 0 \tag{27}$$

where

$$O(V)\equiv (U(V)+s-\beta\overline{U})^{\omega}(V-V_n(s))^{1-\omega}. \tag{28}$$

We first prove existence. Notice first that the constraint $V\in\Phi$ is not binding. To show this, notice $V_n(s)\geq V_n(0)$, then use Proposition 3 and use the observation

$$V_n(0)=u(0)-\phi(0)+\beta\Delta v(0).$$

Notice next that since $U(V)\to-\infty$ as $V\to V_{max}$, the constraint $V\geq V_n(s)$ can be replaced by $V_n(s)\leq V\leq M$ for some sufficiently large M.

Since $U(V)$ is continuous, we have that the constraint set of problem (27), which can now be written as $\{V\in\mathbb{R}:\ V_n(s)\leq V\leq M,\ U(V)+s-\beta\overline{U}\geq 0\}$, is closed and bounded, and hence compact. Since the objective function $O(V)$ is continuous, a solution to problem (27) exists.

We now prove uniqueness. This takes 5 steps.

(i) Notice first that V is not optimal if $V < V^*$, where V^* is defined in equation (22). To show this, suppose $V \in V^*$. Then $V' = V + \epsilon$ could make both the firm and the worker strictly better off, for a positive but sufficiently small ϵ; a contradiction.

(ii) Notice next that since $U(V)$ is concave by Assumption 2, $U(V)$ and hence $U(V) + s - \beta\overline{U}$ are strictly decreasing over $[V^*, v(\infty)/(1-\beta\Delta))$. Since $s \in S_A$ requires $U(V^*) + s - \beta\overline{U} \geq 0$, the equation $U(V) + s - \beta\overline{U} = 0$ has a unique solution. Denote it $\overline{V}(s)$. This allows us to rewrite the constraint $U(V) + s \geq \beta\overline{U}$ as

$$V \leq \overline{V}(s).$$

(iii) Notice next then

$$V \in \{V' \in \mathbb{R} : \; V_n(s) \leq V' \leq M, \; U(V') + s - \beta\overline{U} \geq 0\}$$

if and only if

$$V_n(s) \leq V \leq \overline{V}(s),$$

where it must hold that $V_n(s) \leq \overline{V}(s)$ since the feasibility set cannot be empty. Now suppose $V_n(s) = \overline{V}(s)$. Then of course there is a unique solution that maximizes $O(V)$. In the following we show that the solution to the bargaining problem is unique also in the case $V_n(s) < \overline{V}(s)$.

(iv) So suppose $V_n(s) < \overline{V}(s)$. Notice first that a solution must satisfy either $V = V_n(s)$ or $V = \overline{V}(s)$, or $O'(V) = 0$.

Notice that $V = V_n(s)$ cannot be optimal, because $V' = V_n(s) + \epsilon$ with ϵ positive but sufficiently small can attain $O(V') > 0 = O(V)$. (Note it doesn't matter whether $V_n(s) \geq V^*$ or otherwise.)

Notice that $V = \overline{V}(s)$ cannot be optimal either, because $V' = V_n(s) - \epsilon$ with ϵ positive but sufficiently small can attain $O(V') > 0 = O(V)$.

Therefore, any solution V to problem (27) must satisfy $O'(V) = 0$, or

$$-U'(V)\frac{V - V_n(s)}{U(V) + s - \beta\overline{U}} = \frac{1-\omega}{\omega}. \tag{29}$$

(v) Observe first that in order for (29) to have a solution, it must hold that $U'(V) < 0$, otherwise the left hand side of the equation is non-positive while the right hand side is strictly positive. Thus, wee need only consider the set of Vs over which the value function $U(V)$ is strictly decreasing. Given that $U(V)$ is concave, this in turn implies that the left hand side of (29) is strictly increasing in V over the set of Vs that could potentially solve (29). It then follows that there at most one $V = V_m(s)$ that satisfies (29). Q.E.D. [6]

[6] We have proved (ii) under the assumption that the value function U is differentiable. A proof that does not rely on the differentiability of U is in the appendix.

Proof of Lemma 5

Suppose $0 \notin S_A$. That is, suppose $0 \in S_I$. Then $v(0) = V_n(0) = V_{min} = \frac{u(0)-\phi(0)}{1-\beta\Delta}$.

Consider the following contract: it is σ_0 for the first period, and then the worker is given $s = 0$ to leave the firm. This contract delivers an expected utility equal to $V_0 + \beta\Delta v(0)$ to the worker. Clearly $V_0 + \beta\Delta v(0) \geq V_n(0)$ and $V_0 + \beta\Delta v(0) \in \Phi$. This contract gives an expected profit equal to $\Pi(V_0) + \beta\overline{U}$ to firm. Now $U(V_0 + \beta\Delta v(0)) \geq \Pi(V_0) + \beta\overline{U} \geq \beta\overline{U}$. So $s = 0 \in S_A$. A contraction. Q.E.D.

Proof of Lemma 6 Let v be well defined and continuous. Let $s_2 > s_1 \geq 0$.

(i) That $V_n(s)$ is well defined and continuous is because the objective function is continuous the constraint correspondence is compact. Use then the theorem of the maximum. To show that $V_n(s)$ is strictly increasing in s, notice that with s_2, the worker can always choose to have strictly more consumption in the current period while setting his future assets equal to that with s_1.

(ii) We show that the function $V_m(s)$ is also continuous. This is the case because: (a) The objective function in (17) is continuous in V. (b) Given $U(V) \to -\infty$ as $V \to V_{max}$, there is some $M > 0$ sufficiently large such that for each $s \in S_A$, the constraint $V \geq V_n(s)$ can be replaced by $V_n(s) \leq V \leq M$. This implies a constraint correspondence that is compact valued and continuous. (c) Apply the theorem of the maximum.

We next show that V_m is an increasing function. Observe first that given $U'(V) < 0$ at the optimal V, and $V_n(s)$ is increasing in s, the left hand side of (29) is strictly decreasing in s. Remember we have already shown that the left hand side of (29) is increasing in V. So $V_m(s)$ must be increasing in s.

(iii) If $s_1, s_2 \in S_A$ or $s_1, s_2 \in S_I$, then $v(s_2) \geq v(s_1)$ follows directly right from (i) and (ii). Suppose $s_1 \in S_I$ but $s_2 \in S_A$. Then

$$v(s_2) = \rho V_m(s_2) + (1-\rho)V_n(s_2) \geq V_n(s_2) \geq V_n(s_1) = v(s_1).$$

Suppose $s_1 \in S_A$ but $s_2 \in S_I$. Suppose $v(s_1) > v(s_2)$. Then

$$\rho V_m(s_1) + (1-\rho)V_n(s_1) \geq V_n(s_2),$$

which in turn implies $V_m(s_1) \geq V_n(s_2)$. This contradicts $s_1 \in S_I$ since

$$U(V_m(s_1)) + s_2 - \beta\overline{U} > U(V_m(s_1)) + s_1 - \beta\overline{U} \geq 0.$$

Finally, since $\rho < 1$, v is strictly increasing on $\mathbb{R}_+$. Q.E.D.

Proof of Theorem 7 1. Let (Y, d) denote the space of all bounded and continuous functions $f : \mathbb{R}_+ \to X$ under the *sup* norm, denoted d. (Note that boundedness is needed for $\mathbb{R}_+$ is not compact.) Y is a complete normed vector space.

2. Define a mapping Γ as follows:

$$\forall v \in Y \text{ and } \forall\, s \in \mathbb{R}_+,\ \Gamma(v)(s) = \begin{cases} \rho V_m(s) + (1-\rho)V_n(s), \text{ if } s \in S_A \\ V_n(s), \text{ if } s \in S_I \end{cases} \tag{30}$$

subject to (14)-(17).

Notice that given Lemma 6, the function $\Gamma(v)$ is well defined for all $v \in Y$.

3. We show that Γ maps from Y to Y, that is, $\Gamma : Y \to Y$. We must show that Γ preserves boundedness and continuity. That Γ preserves boundedness is obvious. We now show that Γ preserves continuity. Let $v \in Y$.

(3a) From Lemma 5, we know that the function $V_n(s)$ is continuous and strictly increasing on $\mathbb{R}_+$. We also know that the function $V_m(s)$ is continuous and increasing on S_A.

(3b) Observe that S_I is an open set in $\mathbb{R}_+$ and hence S_A is closed. To show this, let $s \in S_I$. Since $0 \in S_A$ by Lemma 4, we have $s > 0$. This implies $U(V) + s - \beta\overline{U} < 0$ for all $V \in \Phi$ such that $V \geq V_n(s)$. Given the continuity of U and V_n, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subseteq S_I$.

(3c) Since $S_I \in \mathbb{R}_+$ is open, it can be written as an union of disjoint open intervals in $\mathbb{R}_+$.

(3d) Observe next that $[0, (V_n)^{-1}(V^*)] \subseteq S_A$. This is because : $\beta\overline{U} \leq U(V^*)$ by equation (22), the value function $U(V)$ is concave by Assumption 2, and the function $V_n(s)$ is continuous and strictly increasing by Lemma 5.

(3e) With (3c) and (3d), there exists a vector $\{b_0, a_i, b_i,\ i = 1, 2, ..., m\} \subseteq \mathbb{R}_+$ such that

$$S_A = [0, b_0] \cup \left(\bigcup_{i=1}^{m} [a_i, b_i] \right)$$

where

$$(V_n)^{-1}(V^*) \leq b_0 < a_1 \leq b_1 < \cdots < a_m \leq b_m$$

and the values of m and b_m may be infinity.

(3f) Clearly, $\Gamma(v)$ is continuous on $\mathbb{R}^+ \diagdown \{b_0, a_1, b_1, \cdots, a_m, b_m\}$. So, $\Gamma(v)$ is continuous on $\mathbb{R}_+$ if and only if $V_m(s) = V_n(s)$ for $s \in \{b_0, a_1, b_1, \cdots, a_m, b_m\}$.

Suppose $V_m(b_i) > V_n(b_i) \geq V^*$ for $i \in \{0, 1, \cdots, m\}$, Since V_n is continuous, there exists $\varepsilon > 0$ such that $[b_i, b_i + \varepsilon) \subseteq S_A$, a contradiction.

Suppose $V_m(a_i) > V_n(a_i) \geq V^*$ for $i \in \{1, \cdots, m\}$. since $V_n(a_i) > V^*$, U is strictly decreasing for $V \geq V^*$. This implies that $U(V_n(a_i)) + a_i - \beta\overline{U} > 0$. There exists $\varepsilon > 0$ such that $(a_i - \varepsilon, a_i] \subseteq S_a$ which is a contradiction.

We have proved that the function $\Gamma(v)$ is continuous.

4. We show that the mapping Γ is a contraction. Since the underlying space is a normed vector space of bounded and continuous functions, we need only verify that the Blackwell sufficient conditions are satisfied.

(Monotonicity) Let $v_1, v_2 \in Y$ and $v_1 \geq v_2$. We must show that $\Gamma(v_1)(s) \geq \Gamma(v_2)(s)$ for all $s \in \mathbb{R}_+$.

Let S_A^i, S_B^i, V_n^i, V_m^i ($i = 1, 2$) denote the sets S_A and S_B and the functions V_n and V_m induced by v_i through (14)-(17). Notice first that $V_n^1 \geq V_n^2$.

(i) Suppose $s \in S_A^1 \cap S_A^2$. Then the property of the CES objective function guarantees that $V_m^1(s) \geq V_m^2(s)$, and hence $\Gamma(v^1)(s) \geq \Gamma(v^2)(s)$.

(ii) Suppose $s \in S_I^1 \cap S_A^2$. We need only show $V_n^1(s) \geq V_m^2(s)$, which holds, for otherwise $s \in S_A^1$.

(iii) Suppose $s \in S_I^1 \cap S_I^2$. In this case $\Gamma(v^1)(s) = V_n^1(s) \geq V_n^2(s) = \Gamma(v^2)(s)$.

(iv) Suppose $s \in S_A^1 \cap S_I^2$. In this case, $V_m^1(s) \geq V_n^1(s) \geq V_n^2(s)$, implying $s \in S_A^2$, a contradiction. So $s \in S_A^1 \cap S_I^2$ cannot be the case.

We therefore have shown that the mapping Γ is monotonic.

(Discounting) Let $v_1, v_2 \in Y$ and let $v_2 = v_1 + a$ for any $a > 0$. We show that $\Gamma(v^2)(s) \leq \Gamma(v^1)(s) + \beta\Delta a$ for all $s \in \mathbb{R}_+$.

Observe first that $V_n^2(s) = V_n^1(s) + \beta\Delta a$ for all $s \in \mathbb{R}_+$.

Consider first the case $s \in S_A^1 \cap S_A^2$. The desired result in the case holds trivially if the maximized Nash product is zero. In the following, we consider the case where the maximized Nash product is strictly positive.

Let

$$\varphi_i = -\frac{1-\omega}{\omega}\frac{U(V_m^i(s)) + s - \beta\overline{U}}{V_m^i(s) - V_n^i(s)},\ i = 1, 2,$$

where φ_i for $i = 1, 2$ is the slope of the value function $U(V)$ at optimum, i.e., at $V = V_m^i(s)$.

Given the concavity of U and the differentiability of indifference curve, U has to be under the following straight lines

$$f_i(x) = \varphi_i(x - V_m^i(s)) + U(V_m^i(s)),\ i = 1, 2$$

Suppose $V_m^2(s) > V_m^1(s) + \beta\Delta a$. Then $\varphi_1 < \varphi_2 < 0$. Therefore,

$$U(V_m^2(s)) \leq f_1((V_m^2(s))) \Rightarrow U(V_m^1(s)) > f_2((V_m^1(s))),$$

a contradiction. So, we conclude that $V_m^2(s) \leq V_m^1(s) + \beta\Delta a$.

The cases $s \in S_A^1 - S_A^2$, $s \in S_A^2 - S_A^1$, and the case $s \notin S_A^1 \cup S_A^2$ are straightforward to analyze and are left for the reader. This proves that the mapping Γ has the discounting property and hence we have shown that Γ is a contraction.

5. By the contraction mapping theorem then, $v \in Y$ and is the unique fixed point of Γ. So v is continuous and the theorem is proved. Q.E.D.

Proof of Proposition 9 Suppose $i \in \Omega$ but $U(V_i) < \overline{U} - v^{-1}(V_i)$, then move i from Ω to Ω' while not changing the values of a, c_i and V_i. The modified contract would remain feasible, but the firm's value is strictly increased. On the other hand, suppose $U(V_i) > \overline{U} - v^{-1}(V_i)$ but $i \notin \Omega$. Then move i from Ω' to Ω to increase the firm's value. Q.E.D.

Proof of Proposition 10 We prove by way of contradiction. Suppose otherwise. Then there exists a strictly monotonic sequence $\{s_q\}$ such that $s_q \in S_I$ for all q, and $s_q \to 0$ as $q \to \infty$.

Next we show that it must hold that $V_n(s_q) \to 0$ as $q \to \infty$. Since $s_q \in S_I \ \forall q$,

$$v(s_q) = V_n(s_q) = \max_{0 \le c \le s_q} \{u(c) - \phi(0) + \beta\Delta v[(1+r)(s_q - c)/\Delta]\}.$$

Let $q \to 0$ on both sides of the above equation to obtain

$$u(0) - \phi(0) + \beta\Delta v(0) = v(0)$$

or

$$v(0) = [u(0) - \phi(0)]/(1 + \beta\Delta) = 0.$$

So $V_n(s_q) \to v(0) = 0$ as $q \to \infty$.

Now for each q and s_q, consider the following contract for the unemployed worker with assets s_q. The worker is employed for one period. For the period the worker is employed, his compensation is determined by σ_0. The worker is then terminated with assets s. So the worker's utility under this contract is $H_0 + \beta\Delta v(s_q)$. For q large enough, it must then hold that

$$H_0 + \beta\Delta v(s_q) \ge (1 - \beta\Delta)V_n(s_q) + \beta\Delta V_n(s_q) = V_n(s_q).$$

This holds because (a) $s_q \in S_I$ so $v(s_q) = V_n(s_q)$; (b) $H_0 > 0$; and (c) $V_n(s_q) \to 0$ as $q \to \infty$.

Finally, notice that for q large enough, it holds that

$$\Pi(H_0) + s_q - \beta\Delta s_q + \beta\overline{U} > \beta\overline{U}.$$

Thus we have shown $s_q \in S_A$ for q large enough. A contradiction. Q.E.D.

Proof of Proposition 11 That the worker is terminated with expected utility V implies

$$U(V) < \overline{U} - v^{-1}(V). \tag{31}$$

Now this worker would go immediately to the market to look for a new match if and only if

$$\exists\, V' \ge V_n(v^{-1}(V)) \text{ such that } U(V') + v^{-1}(V) \ge \beta\overline{U}.$$

But this implies the existence of $V_m(v^{-1}(V))$ and it holds that $V_m(v^{-1}(V)) \ge V$. Let $V' = V_m(v^{-1}(V))$. Then

$$U(V') + v^{-1}(V) \ge \beta\overline{U}. \tag{32}$$

Suppose $\overline{U} = 0$. Equations (31) and (32) together imply $U(V') > U(V)$. Q.E.D.

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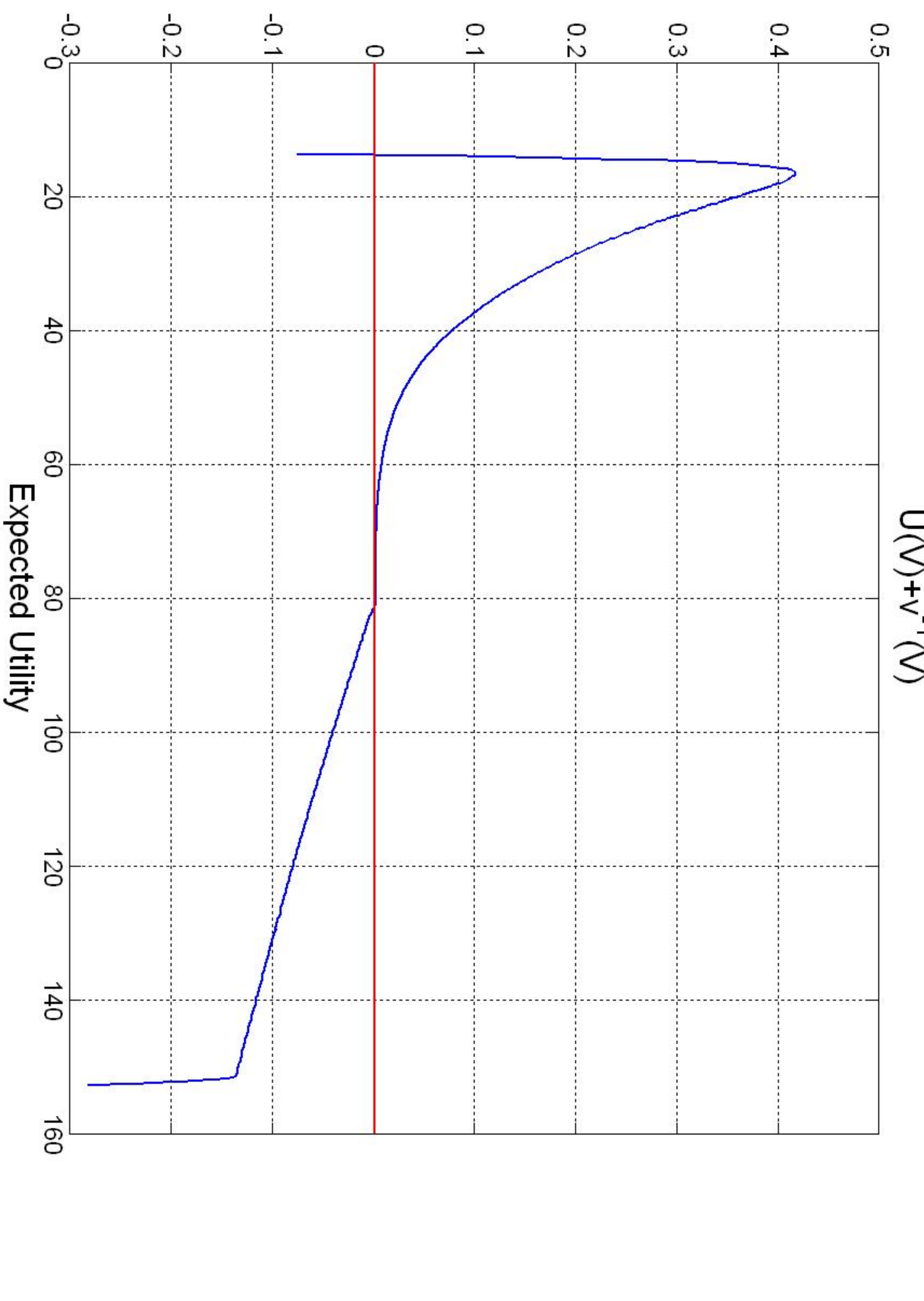

Figure 1: Firm's net gains from retaing the worker

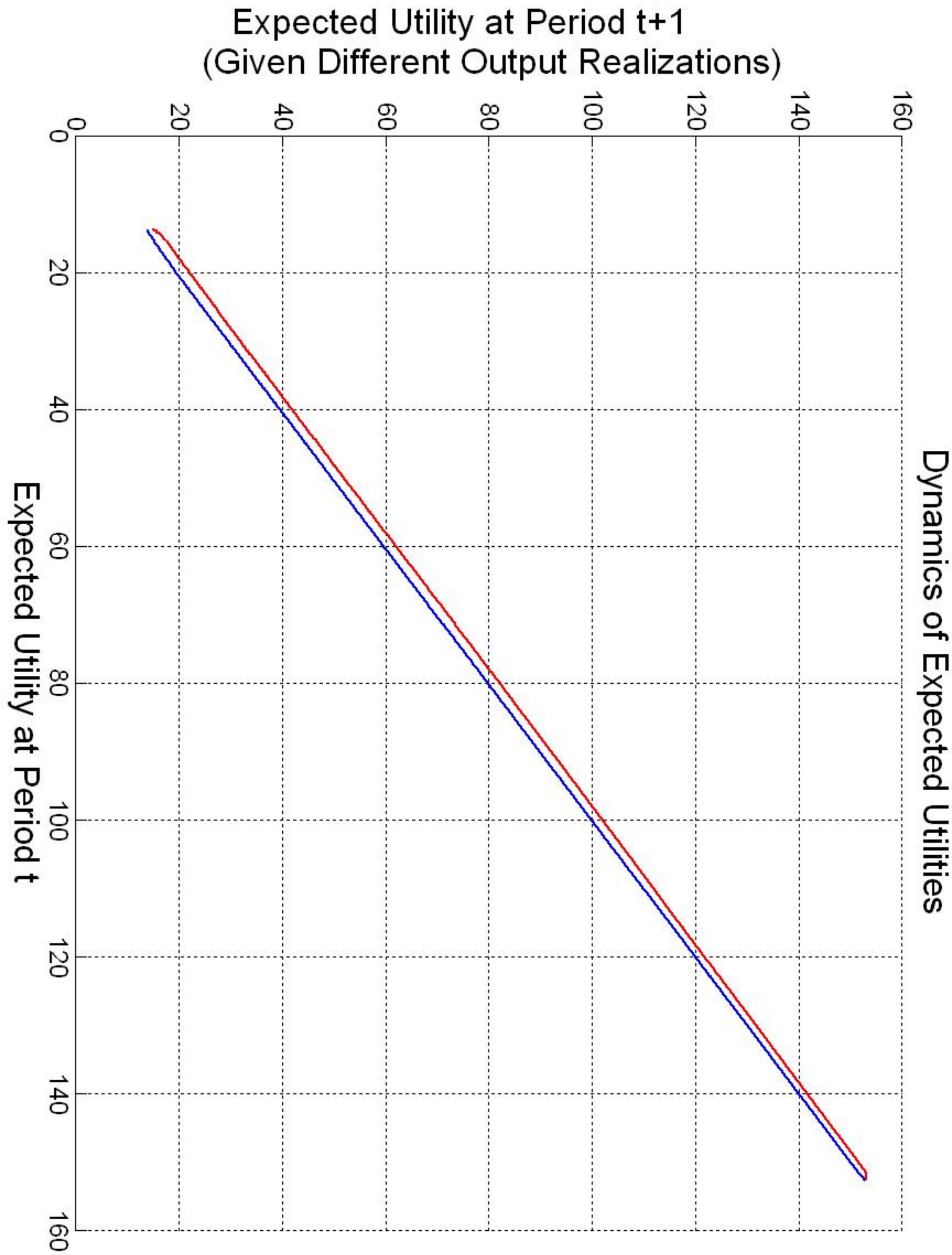

Figure 2: Law of motion for the employed worker's utility

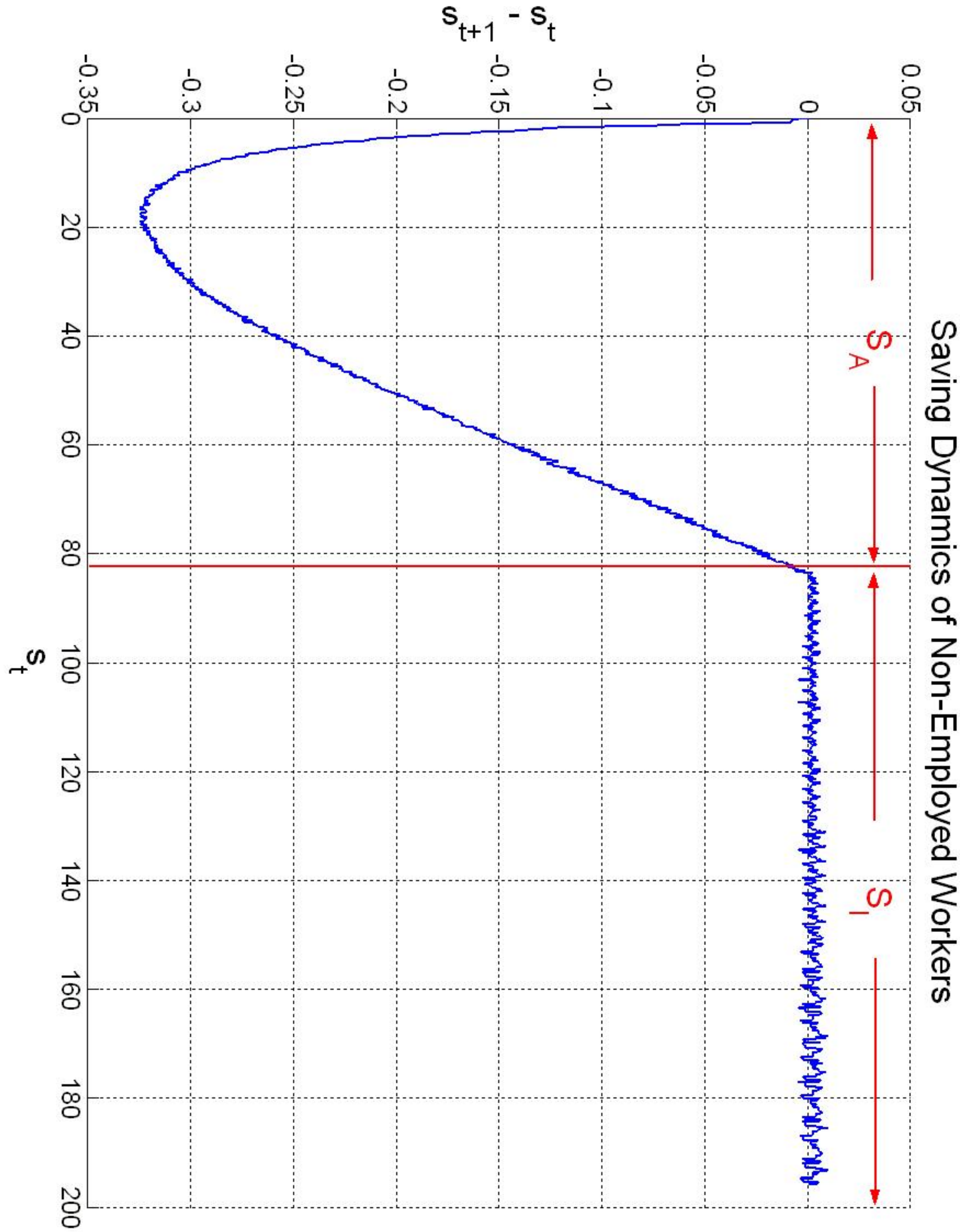

Figure 3: Law of motion for the non-employed worker's assets

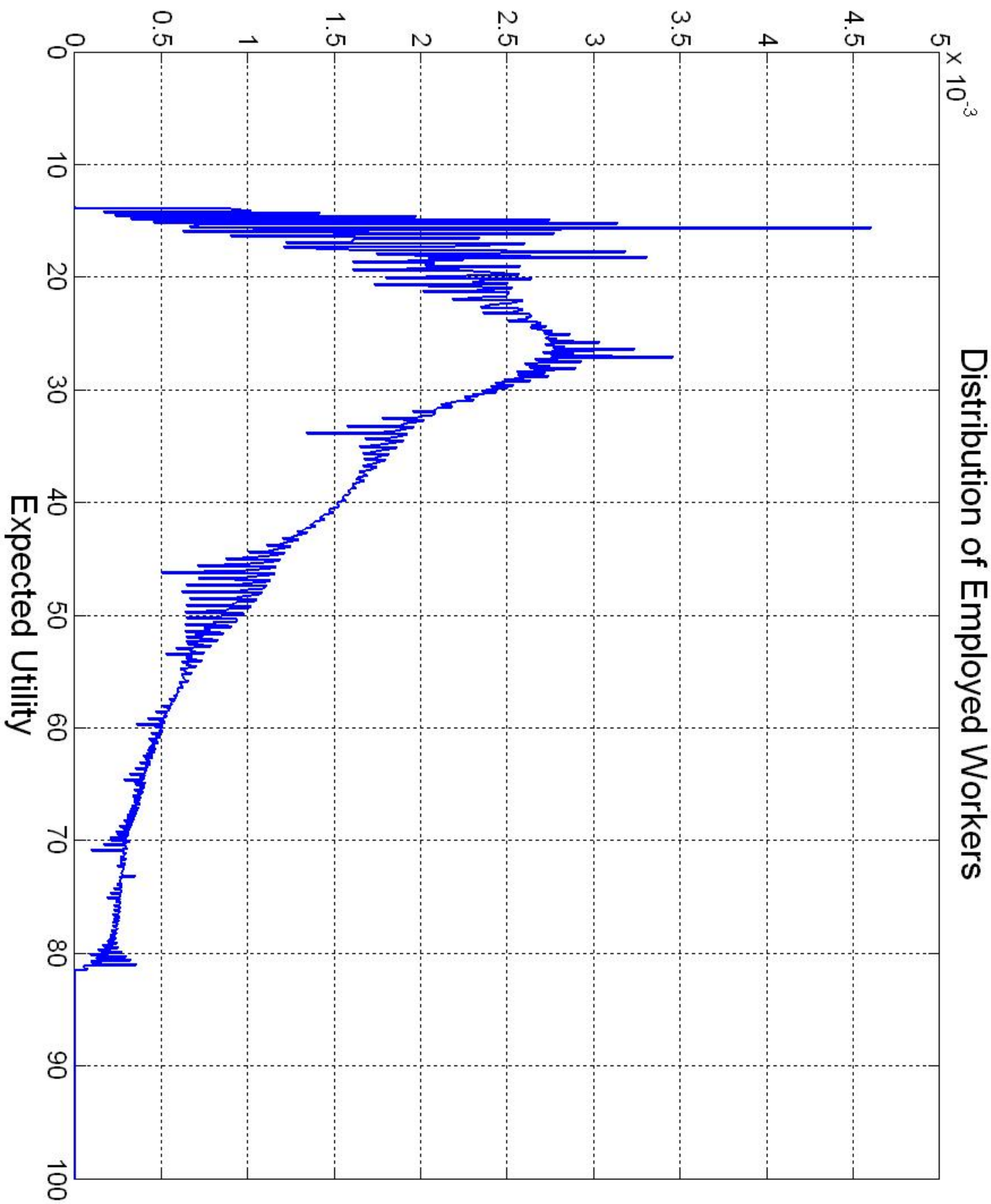

Figure 4: Distribution of employed workers

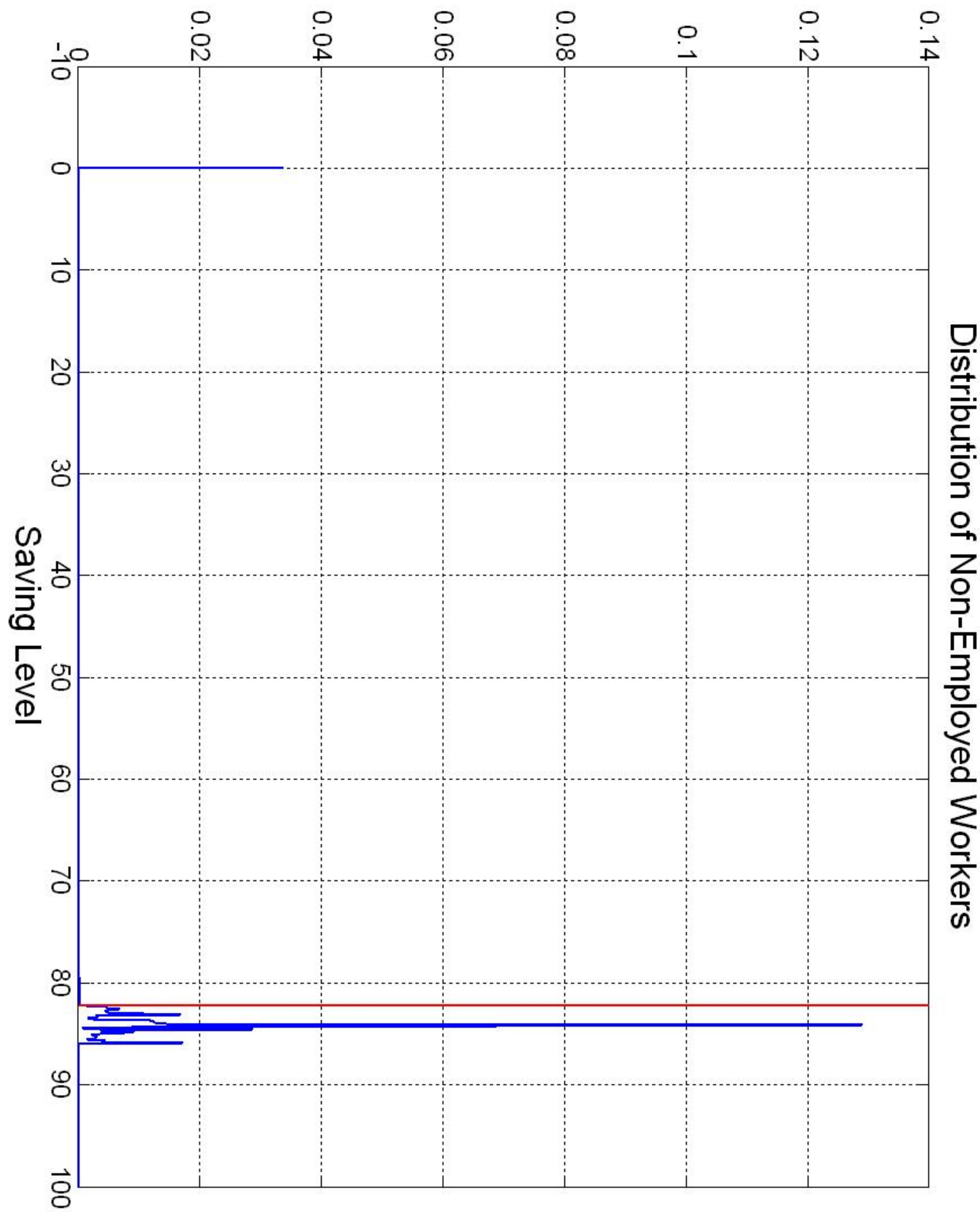

Figure 5: Distribution of non-workers

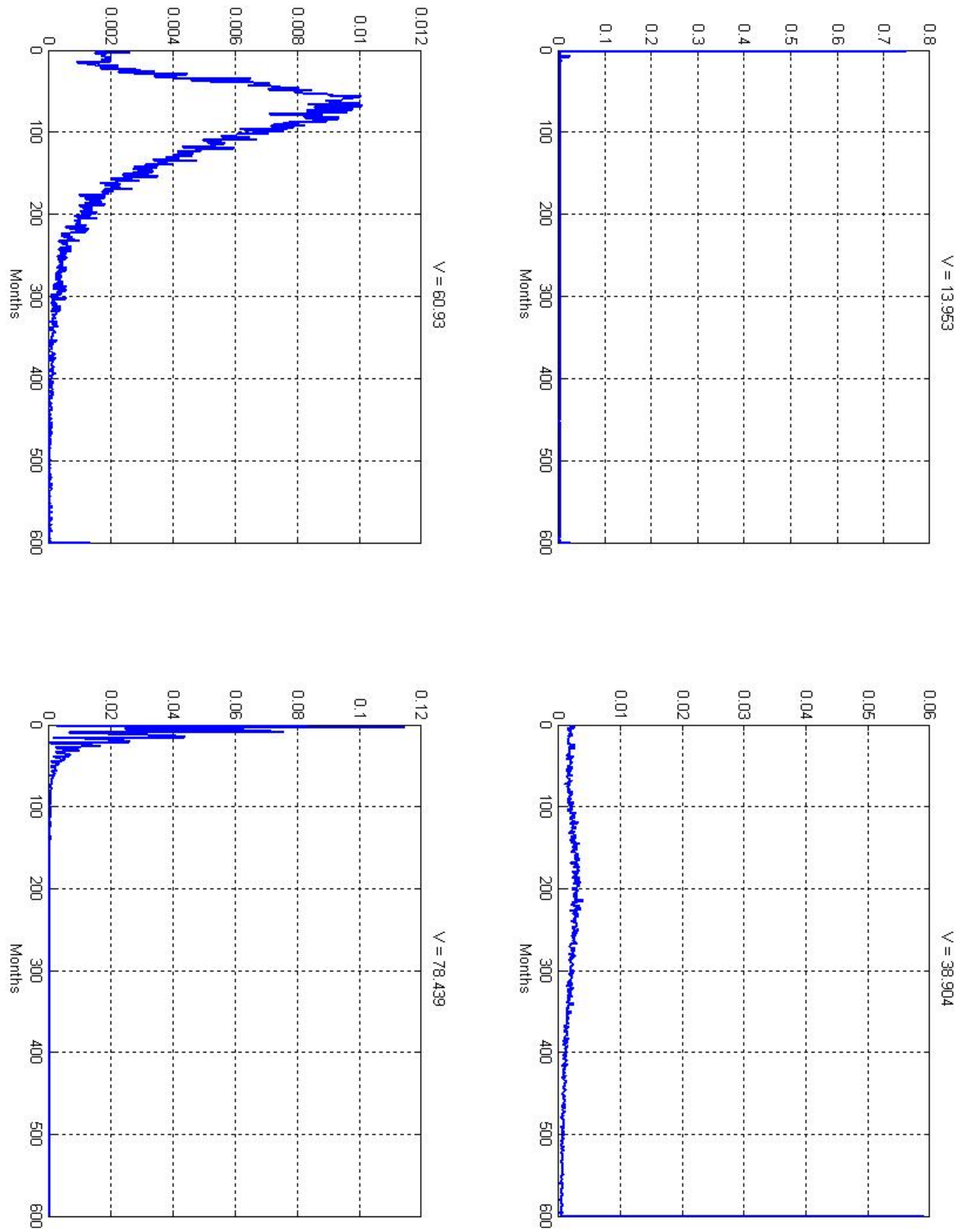

Figure 6: Distribution of employment durations

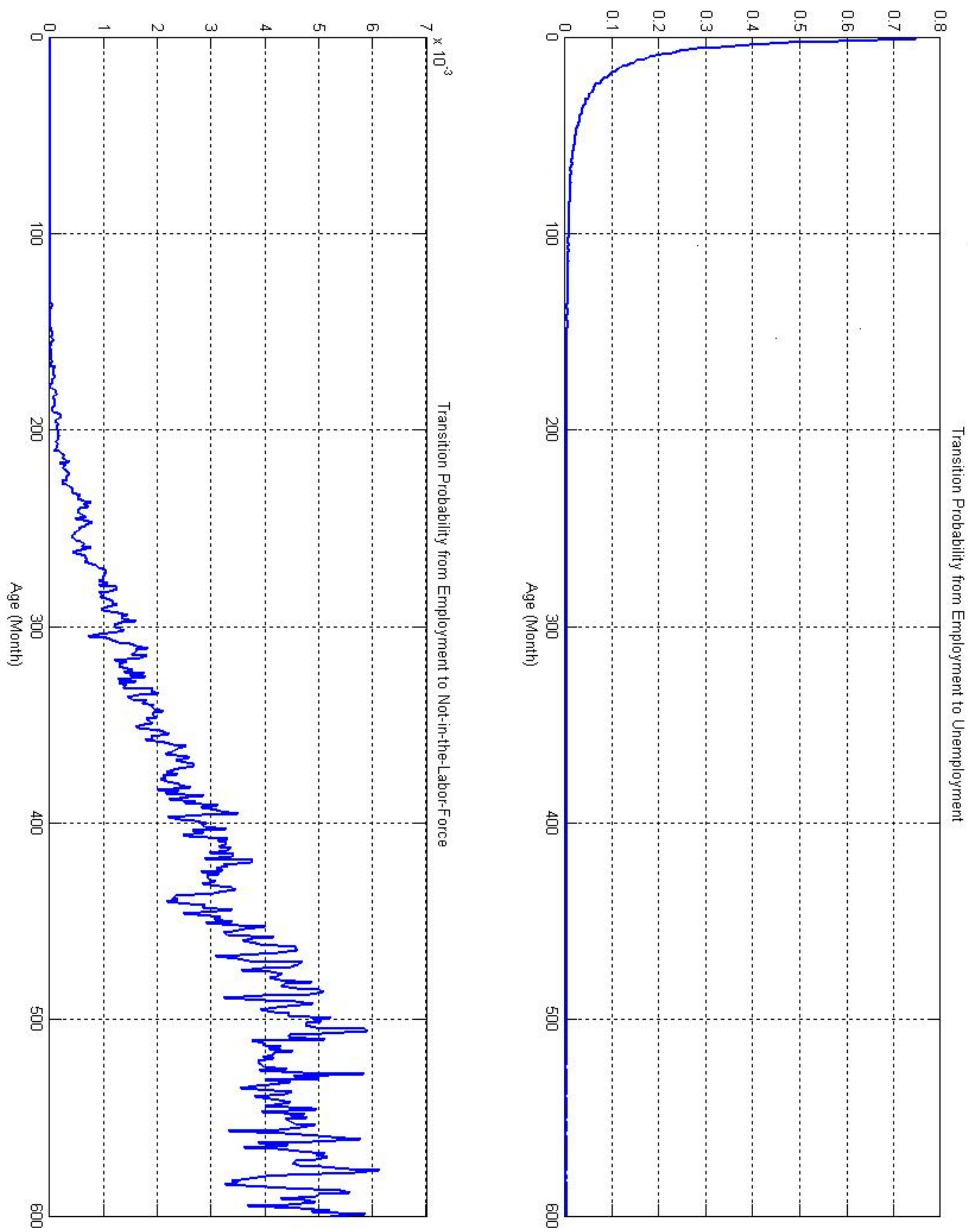

Figure 7: Probabilities of termination

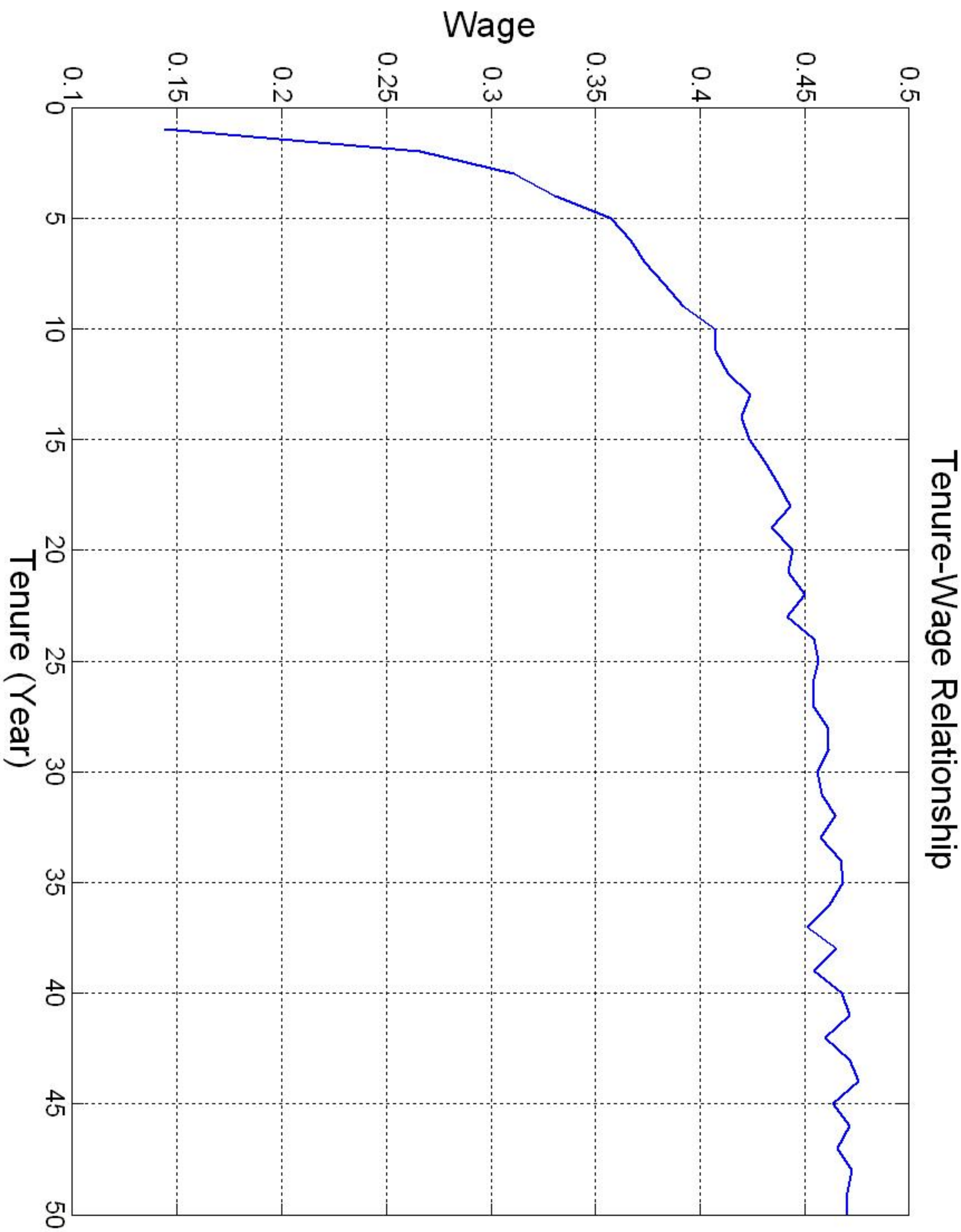

Figure 8: Wage versus tenure

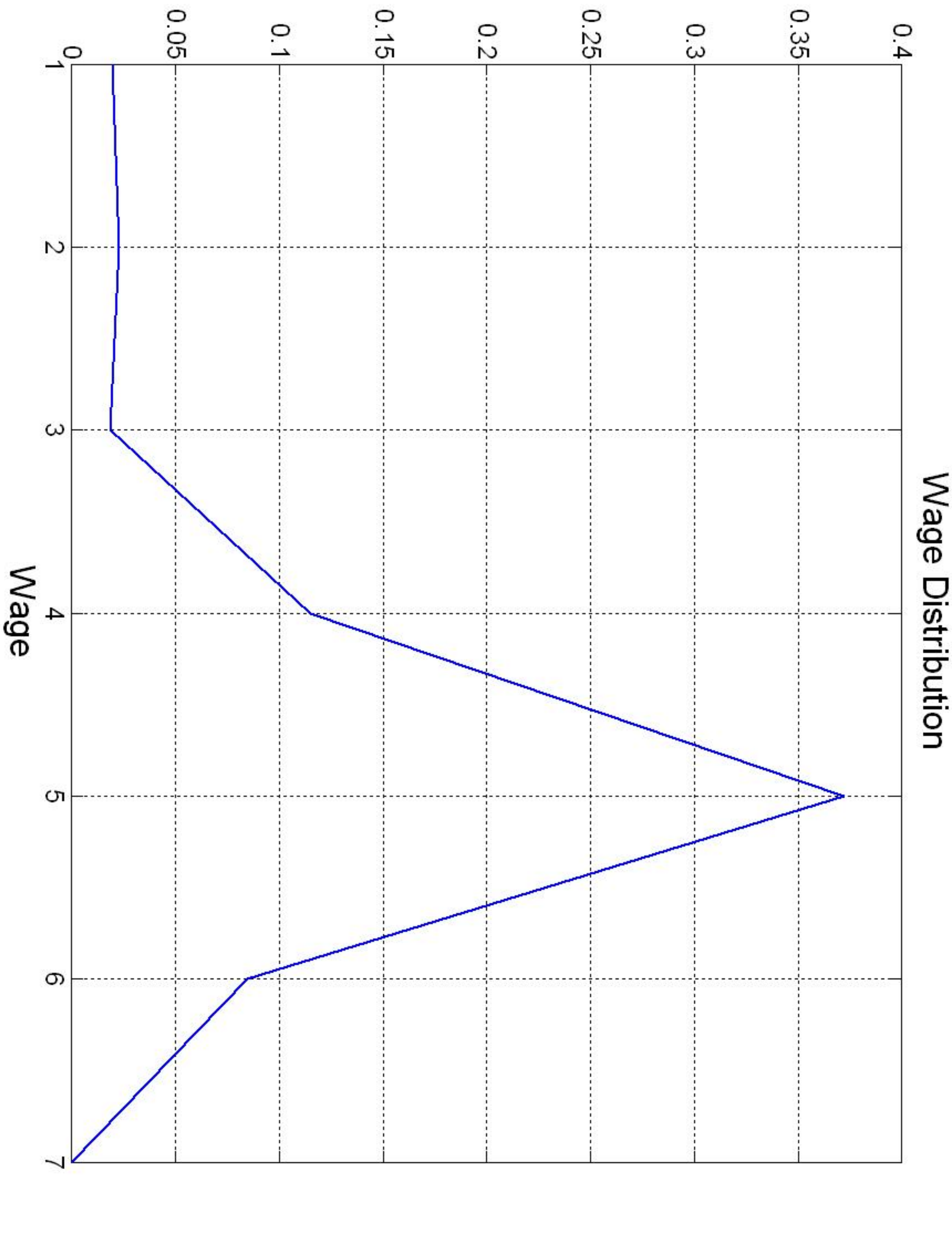

Figure 9: The wage distribution